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Supersymmetric gauge theories in twistor space

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Abstract: We construct a twistor space action for $\mathcal{N} = 4$ super Yang-Mills theory and show that it is equivalent to its four dimensional spacetime counterpart at the level of perturbation theory. We compare our partition function to the original twistor-string proposal, showing that although our theory is closely related to string theory, it is free from conformal supergravity. We also provide twistor actions for gauge theories with $\mathcal{N} < 4$ supersymmetry, and show how matter multiplets may be coupled to the gauge sector.

Keywords: Supersymmetric gauge theory, Duality in Gauge Field Theories.

Contents

1. Introduction

In his construction of twistor-string theory [1], Witten found that the open string sector of the B-model on $\mathbb{CP}^{3|4}$ coincides with that of $\mathcal{N} = 4$ SYM in spacetime. States of the open string are described by a (0,1)-form $\mathcal{A}$ and expanding this in terms of the fermionic directions of $\mathbb{CP}^{3|4}$ yields component fields which constitute an $\mathcal{N} = 4$ multiplet when interpreted via the Penrose transform [2, 3]. Unfortunately, holomorphic Chern-Simons theory (i.e. the open string field theory of the B-model with space-filling D-branes [4]) on $\mathbb{CP}^{3|4}$ only provides the anti-selfdual couplings of the SYM theory. To overcome this, building on Nair's observation [5] that MHV amplitudes are supported on holomorphic degree 1 curves in twistor space, in [1] Witten supplemented the twistorial B-model with D1-instantons, obtaining the missing interactions from the effective theory of D1-D5 strings. However, his procedure leads to multi-trace interactions. Such terms would not be present in connected SYM tree amplitudes, and in [6] they were interpreted as mixing in conformal supergravity, indicating that the D1 instantons prevent the open and closed string sectors from decoupling.

Despite this fundamental problem, twistor-string theory has inspired many remarkable and powerful new approaches to Yang-Mills theory. Following Witten, in [7, 8] Roiban, Spradlin & Volovich were able to 'extract' YM tree amplitudes from the twistor-string by simply discarding the multi-trace terms. Meanwhile, by considering maximally disconnected D1 instantons, Cachazo, Svrček & Witten [9] developed tree-level scattering rules based on using MHV amplitudes with arbitrary numbers of external particles as primary vertices.[1] Using unitarity methods, the MHV diagrams may be tied together to form loops [12] obtaining expressions that agree with $\mathcal{N} = 4$ 1-loop amplitudes in the literature [13]. In particular, conformal supergravity does not arise.

We believe that the success of these results strongly indicates the existence of a theory in twistor space which is exactly equivalent to spacetime $\mathcal{N} = 4$ SYM. In this paper, we verify this by presenting a twistor action and showing explicitly that its partition function coincides with that of the standard spacetime theory at the perturbative level. A twistor construction and action for non-supersymmetric Yang-Mills and conformal gravity appeared in [14] together with a formal argument that attempted to make contact with twistor-string theory — this latter argument, however, was too formal to be sensitive to the issue of multi-trace terms.

Our action consists of two parts; a holomorphic Chern-Simons theory and a term which is closely related, but not identical to Witten's D-instantons. The action is invariant under the full group of complex gauge transformations on twistor space, together with additional twistor cohomological gauge freedom. This gauge freedom consists of free functions of six variables rather than the four variables of spacetime and this extra freedom may either be partially fixed to cast the theory into standard spacetime form, or fixed in a way not accessible from space to cast the theory into a form that makes the MHV diagram formalism transparent. In a companion paper [15], we present a study of perturbation theory based on the MHV form of the action, showing how it may be used in calculating loop amplitudes. (We note that the MHV diagram formalism has been derived at tree level from spacetime considerations in [16, 17]).

The outline of the paper is as follows. In section 2 we begin by reviewing the geometry of twistor superspace, and in particular the reality conditions we employ to descend to Euclidean space. Similar expositions may be found in [1 − 3, 18 − 21], for example. The main results of the paper are contained in section 3, where we present our action and show that it is equivalent to $\mathcal{N} = 4$ SYM at the perturbative level. By breaking the symmetry of the maximally supersymmetric theory, it is also possible to obtain twistor actions for YM theories with $\mathcal{N} < 4$. As we discuss in section 4, this may be done by a method that is similar, but not identical to working on weighted twistor superspaces. When $\mathcal{N} < 4$ additional multiplets are possible and, following Ferber [18], we explain how to construct these and minimally couple them to the gauge theory. One of the most important questions our investigation raises is whether, and if so how, the ideas of this paper are related to string theory. In section 5 we first refine the arguments of [14] to explain precisely how our twistor action differs from Witten's original proposal. Nonetheless, we will conclude by proposing that indeed there are initimate connections with a modification of the twistor-string.

Our conventions are those of Penrose & Rindler [2] (and differ slightly from those of Witten [1]): primed and unprimed capital indices $A', B', C', \ldots$ and $A, B, C, \ldots$ label elements of $\mathbb{S}^+$ and $\mathbb{S}^-$, the left and right spin bundles, respectively. They are contracted

[1] The recursion relations of Britto, Cachazo & Feng [10], though even more succinct, are of less relevance for this paper; they are closer to the theory in ambitwistor space [11].

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using the SL(2)-invariant tensors ϵ_{AB} and $\epsilon^{A'B'}$, with the conventions $\omega \cdot \lambda = \omega^A \lambda_A = \omega^A \lambda^B \epsilon_{BA}$ and $\pi \cdot \mu = \pi_{A'} \mu^{A'} = \pi_{A'} \mu_{B'} \epsilon^{B'A'}$. Roman indices $a, b, c, \ldots$ from the beginning of the alphabet denote elements of the tangent (or cotangent) bundles to four-dimensional Euclidean space $\mathbb{E}$. These three sets of indices can be viewed as 'abstract' in the sense that $\mathrm{d}x^a$ is a particular 1-form, rather than the a^{th} component of $\mathrm{d}x$. This allows us to write $\mathrm{d}x^a = \mathrm{d}x^{AA'}$ since the isomorphism $T^*\mathbb{E} \simeq \mathbb{S}^+ \otimes \mathbb{S}^-$ tells us they are the same geometrical object. If components are required, they may be obtained using the standard van der Waerden symbols $\sigma^a_{BB'}$. Greek indices $\alpha, \beta, \gamma, \ldots$ denote elements of the (co-)tangent spaces to $\mathbb{C}^4$ while Roman indices $i, j, k, \ldots$ from the middle of the alphabet ranging from 1 to $\mathcal{N}$ label the fermionic directions.

2. The geometry of twistor superspaces

The projective twistor space of complexified, compactified, flat spacetime is $\mathbb{CP}^3$. In this paper we will be concerned with the associated superspaces $\mathbb{CP}^{3|\mathcal{N}}$ and their relation to superspacetimes. To obtain interesting cohomology (and, physically, to have some notion of an asymptotic region for in and out states) one must work on the non-compact space $\mathbb{PT}^{3|\mathcal{N}}$ obtained by removing a $\mathbb{CP}^{1|\mathcal{N}}$ (corresponding to the lightcone 'at infinity' in spacetime) from $\mathbb{CP}^{3|\mathcal{N}}$. $\mathbb{CP}^{3|\mathcal{N}}$ may be provided with homogeneous coordinates $[Z^\alpha, \psi^i] = [\omega^A, \pi_{A'}, \psi^i]$, defined as always with respect to the equivalence relation $(Z^\alpha, \psi^i) \sim (tZ^\alpha, t\psi^i)$ where $t \in \mathbb{C}^*$. It is then convenient to remove the $\mathbb{CP}^{1|\mathcal{N}}$ whose coordinates are $[\omega^A, 0, \psi^i]$.

There are two spaces of immediate interest: the space of holomorphic lines $\mathbb{CP}^{1|0} \hookrightarrow \mathbb{PT}^{3|\mathcal{N}}$ and the space of holomorphic superlines $\mathbb{CP}^{1|\mathcal{N}} \hookrightarrow \mathbb{PT}^{3|\mathcal{N}}$. A holomorphic line is a $\mathbb{CP}^{1|0}$ linearly embedded in $\mathbb{PT}^{3|\mathcal{N}}$ as

$$\omega^A = x_-^{AA'} \pi_{A'} \qquad \psi^i = \widetilde{\theta}^{A'i} \pi_{A'}, \tag{2.1}$$

and hence is parametrized by the 4+2$\mathcal{N}$ complex coefficients $(x_-^{AA'}, \widetilde{\theta}^{A'i})$. On the other hand, a holomorphic superline in $\mathbb{PT}^{3|\mathcal{N}}$ is a $\mathbb{CP}^{1|\mathcal{N}}$ linearly embedded via

$$\omega^A = x_+^{AA'} \pi_{A'} - \theta_i^A \psi^i \tag{2.2}$$

and so is parametrized by the 4+2$\mathcal{N}$ different complex coefficients $(x_+^{AA'}, \theta_i^A)$. It is important to note that $[\pi_{A'}]$ are the only independent variables on the $\mathbb{CP}^{1|0}$ of (2.1), while in (2.2) the independent variables on the $\mathbb{CP}^{1|\mathcal{N}}$ are $[\pi_{A'}, \psi^i]$. We also stress that θ and $\widetilde{\theta}$ are independent complex fermionic parameters, in particular they are not related by complex conjugation.

Equations (2.1) and (2.2) show that the $\mathbb{CP}^{1|0}$ intersects the $\mathbb{CP}^{1|\mathcal{N}}$ whenever

$$\left(x_+^{AA'} - x_-^{AA'} - 2\theta_i^A \widetilde{\theta}^{A'i}\right) \pi_{A'} = 0. \tag{2.3}$$

If the contents of the brackets themselves vanish, then this is true for all $[\pi_{A'}]$ and hence the $\mathbb{CP}^{1|0}$ lies entirely within the $\mathbb{CP}^{1|4}$. Thus the space $\mathcal{M}^{4|4\mathcal{N}}$ of lines inside superlines inside twistor superspace is complexified superspacetime, with $4+4\mathcal{N}$ complex co-ordinates $(x^{AA'},\ \theta^A_i,\ \widetilde{\theta}^{A'i})$ where

$$\begin{aligned} x_+^{AA'} &= \ x^{AA'} + \theta_i^A\widetilde{\theta}^{A'i} \\ x_-^{AA'} &= \ x^{AA'} - \theta_i^A\widetilde{\theta}^{A'i}. \end{aligned} \tag{2.4}$$

These equations identify (x_+,θ) and $(x_-,\widetilde{\theta})$ as coordinates on complexified chiral and anti-chiral superspace, respectively.

Throughout most of this paper, we shall be interested in four-dimensional Euclidean spacetime $\mathbb{E}$ and its associated superspaces. In Euclidean signature, complex conjugation sends primed spinors to primed spinors and unprimed to unprimed by the formulae

$$\omega^A \to \hat{\omega}^A = (-\overline{\omega^1}, \overline{\omega^0}) \qquad \text{and} \qquad \pi_{A'} \to \hat{\pi}_{A'} = (-\overline{\pi_{1'}}, \overline{\pi_{0'}})\,. \tag{2.5}$$

These satisfy $\hat{\hat{\omega}}^A = -\omega^A$ and $\hat{\hat{\pi}}_{A'} = -\pi_{A'}$, so there are no non-vanishing real spinors. The conjugation induces SU(2) invariant inner products $\|\omega\|^2 = \omega^A\hat{\omega}_A$ and $\|\pi\|^2 = \pi_{A'}\hat{\pi}^{A'}$ on $\mathbb{S}^-$ and $\mathbb{S}^+$ respectively, and is extended to twistor space in the obvious way

$$Z^\alpha = (\omega^A, \pi_{A'}) \to \hat{Z}^\alpha = (\hat{\omega}^A, \hat{\pi}_{A'}) \tag{2.6}$$

which again has no fixed points in the projective space. The conjugation (2.6) provides twistor space with a non-holomorphic fibration over S^4. In the open region $\mathbb{PT}$ this is a fibration over $\mathbb{E}^4$ given by

$$Z^\alpha = (\omega^A, \pi_{A'}) \to x^{AA'} = \frac{\omega^A\hat{\pi}^{A'} - \hat{\omega}^A\pi^{A'}}{\|\pi\|^2}. \tag{2.7}$$

It follows from (2.5) that $x^{AA'}$ is real, and it is easy to check that $\det(x^{AA'}) = g(x,x)$ where g is the standard Euclidean metric on $\mathbb{R}^4$.

One may extend the conjugation (2.5) to the $\mathcal{N}$ fermionic directions by defining an antiholomorphic map $s : \mathbb{CP}^{3|\mathcal{N}} \to \mathbb{CP}^{3|\overline{\mathcal{N}}}$. Our notation $\overline{\mathcal{N}}$ here indicates that s is not an involution. Generically, s maps the twistor superspace to a different superspace that has the same body, but whose fermionic directions are in the complex conjugate representation of the R-symmetry group. If the R-symmetry group admits quaternionic representations (i.e. when $\mathcal{N}=2$ or $\mathcal{N}=4$ and the R-symmetry groups are SU(2) or SP(2) $\subset$ SU(4)), then s may be involutive on the whole superspace. For example, when $\mathcal{N}=4$ one may define $s : \mathbb{CP}^{3|4} \to \mathbb{CP}^{3|4}$ by

$$s([Z^\alpha, \psi^i]) = [\hat{Z}^\alpha, \hat{\psi}^{\bar{\imath}}] = [\hat{\omega}^A, \hat{\pi}_{A'}, -\overline{\psi^2}, \overline{\psi^1}, -\overline{\psi^4}, \overline{\psi^3}]. \tag{2.8}$$

In such cases, one can extend the fibration (2.7) to the $\mathcal{N}=2,4$ superspaces by imposing reality conditions on the fermions $\widetilde{\theta}^{A'i}$. One finds $\mathbb{PT}^{3|4} \to \mathbb{E}_-^{4|8}$ and $\mathbb{PT}^{3|2} \to \mathbb{E}_-^{4|4}$, where the projection is given by

$$x_-^{AA'} = \frac{\omega^A\hat{\pi}^{A'} - \hat{\omega}^A\pi^{A'}}{\|\pi\|^2} \qquad \text{and} \qquad \widetilde{\theta}^{A'i} = \frac{\psi^i\hat{\pi}^{A'} - \hat{\psi}^i\pi^{A'}}{\|\pi\|^2} \tag{2.9}$$

and both $x^{AA'}$ and $\widetilde{\theta}^{A'i}$ are real under s. However, when $\mathcal{N}=1$ no such reality conditions may be imposed without forcing the fermions to vanish. (This is a well-known irritation in discussing Euclidean supersymmetry; see e.g. [22] for a discussion.) Even when no reality conditions are imposed on the fields, the fibration (2.9) is still useful as it induces a natural choice of real contour in the space of $\mathbb{CP}^{1|0}$s.

To make use of all this, in particular the fibration (2.7) of the body $\mathbb{PT}$ over $\mathbb{E}$, it proves convenient to work with non-holomorphic coordinates $(x_{-}^{AA'},[\pi_{A'}])$ on $\mathbb{PT}$, where the scale of $\pi_{A'}$ is projected out. These coordinates provide a basis for (0,1)-forms which we write as

$$\bar{e}^0=\frac{\hat{\pi}^{A'}\mathrm{d}\hat{\pi}_{A'}}{(\pi\cdot\hat{\pi})^2}\qquad\qquad \bar{e}^A=\frac{\mathrm{d}x_{-}^{AA'}\hat{\pi}_{A'}}{\pi\cdot\hat{\pi}},\tag{2.10}$$

where the factors of $\pi\cdot\hat{\pi}$ are included for later convenience and ensure that the basis forms only have holomorphic weight. The frame is dual to (0,1)-vectors

$$\overline{\partial}_0=(\pi\cdot\hat{\pi})\pi_{A'}\frac{\partial}{\partial\hat{\pi}_{A'}}\qquad\qquad \overline{\partial}_A=\pi^{A'}\frac{\partial}{\partial x_{-}^{AA'}}\tag{2.11}$$

in the sense that $\overline{\partial}_0\lrcorner\,\bar{e}^0=1$, $\overline{\partial}_A\lrcorner\,\bar{e}^B=\delta_A^B$, $\overline{\partial}_0\lrcorner\,\bar{e}^A=\overline{\partial}_A\lrcorner\,\bar{e}^0=0$ and the $\overline{\partial}$-operator is expressed as $\overline{\partial}=\bar{e}^0\overline{\partial}_0+\bar{e}^A\overline{\partial}_A$. Note also that

$$\Omega:=\frac{1}{4!}\epsilon_{\alpha\beta\gamma\delta}Z^\alpha\mathrm{d}Z^\beta\wedge\mathrm{d}Z^\gamma\wedge\mathrm{d}Z^\delta=(\pi\cdot\hat{\pi})^4\,e^0\wedge e^A\wedge e_A\tag{2.12}$$

where e^0 and e^A are the complex conjugates of (2.10). This basis will be helpful when we relate the twistor and spacetime SYM actions in the next section using methods that are described in detail by Woodhouse [19].

3. $\mathcal{N}=4$ SYM on twistor space

In this section, we will construct a twistorial action for the full $\mathcal{N}$=4 SYM theory. Our theory manifestly contains only single-trace amplitudes for connected diagrams at tree-level and is thus free from conformal supergravity. We will see that different gauge choices allow the twistor action to interpolate between the usual spacetime $\mathcal{N}=4$ theory and a Lagrangian directly adapted to the MHV diagram formalism.

An $\mathcal{N}=4$ gauge multiplet is CPT self-conjugate and may be represented on twistor superspace as an element $\mathcal{A}\in\Omega^{(0,1)}_{\mathbb{PT}^{3|4}}(\mathrm{End}(E))$ where $\Omega^{(p,q)}_{\mathbb{PT}^{3|4}}$ denotes the space of smooth (not necessarily holomorphic) (p,q)-forms on $\mathbb{PT}^{3|4}$ and $E\to\mathbb{PT}$ is a vector bundle whose structure group is the complexification of the spacetime gauge group. We follow Witten [1] in assuming[2] both that $\mathcal{A}$ has only holomorphic dependence on ψ, and that $\partial/\partial\hat{\psi}^{\bar{\imath}}\lrcorner\,\mathcal{A}=0$. To fix notation, the ψ-expansion of $\mathcal{A}$ is taken to be

$$\mathcal{A}=a+\psi^i\lambda_i+\frac{1}{2!}\psi^i\psi^j\phi_{ij}+\frac{\epsilon_{ijkl}}{3!}\psi^i\psi^j\psi^k\chi^l+\frac{\epsilon_{ijkl}}{4!}\psi^i\psi^j\psi^k\psi^l b,\tag{3.1}$$

[2]These assumptions amount to taking as a starting point the Čech cohomology with repect to an open cover $\{U_i\}$ of $\mathbb{CP}^{3|4}$, obtained by pulling back a cover of $\mathbb{CP}^3$ using a fibration $\mathbb{CP}^{3|4}\to\mathbb{CP}^3$. A Dolbeault representative $\mathcal{A}$ can be constructed from a Čech representative $\mathcal{A}_{ij}$ defined on $U_i\cap U_j$ by the formula $\mathcal{A}=\sum_j\mathcal{A}_{ij}\bar{\partial}\rho_j$, where ρ_i is a partition of unity pulled back from $\mathbb{CP}^3$ and subordinate to U_i. We leave it to the reader to check that this has the right properties.

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where $\{a, \lambda_i, \phi_{ij}, \chi^i, b\}$ are smooth (0,1)-forms on $\mathbb{PT}$ of weight $\{0, -1, -2, -3, -4\}$ respectively.

3.1 The twistorial $\mathcal{N} = 4$ action

Our twistor action $S[\mathcal{A}]$ will be expressed as a sum $S[\mathcal{A}] = S_1[\mathcal{A}] + S_2[\mathcal{A}]$ as follows. The kinetic terms and anti-selfdual interactions of $\mathcal{N} = 4$ SYM theory may be described by a holomorphic Chern-Simons theory on $\mathbb{PT}^{3|4}$ with action [1]

$$S_1[\mathcal{A}] = \frac{\mathrm{i}}{2\pi} \int \Omega \, \mathrm{d}^4\psi \wedge \mathrm{tr} \left\{ \mathcal{A} \wedge \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right\} \tag{3.2}$$

where tr indicates a trace using a Killing form on E (and involves a choice of Hermitian metric on the fibres). Ω was defined in equation (2.12) and $\Omega \, \mathrm{d}^4\psi$ is a holomorphic volume form on the superspace. The action is invariant under gauge transformations

$$\bar{\partial} + \mathcal{A} \to g(\bar{\partial} + \mathcal{A})g^{-1} \tag{3.3}$$

where g is an $\mathrm{SU}(N)$-valued section of $E \to \mathbb{PT}^{3|4}$ that is homotopic to the identity, and we require $g \to 1$ asymptotically. This is a considerably greater freedom than in spacetime because g, like $\mathcal{A}$, is only required to be smooth and so is a function of six variables. Ordinarily, one chooses the numerical coefficient of the Chern-Simons action on a real 3-manifold M to ensure that the partition function is invariant under gauge transformations g that do not map to the identity in $\pi_3(G)$. The issue does not arise here because, on twistor (super)space, $b_3 = 0$ and the normalization of (3.2) is arbitrary. S_1 leads to an equation of motion $\bar{\partial}\mathcal{A} + [\mathcal{A}, \mathcal{A}] = 0$ which implies that the (0,2)-component of the curvature of $\mathcal{A}$ vanishes. Hence, on-shell $\bar{\partial}_{\mathcal{A}}$ defines an integrable complex structure on E. E is then holomorphic, and so describes an anti-selfdual solution of the $\mathcal{N} = 4$ SYM equations in complex spacetime via the Penrose-Ward correspondence (see e.g. [2, 3]).

To obtain the full $\mathcal{N} = 4$ SYM theory, one considers the action $S = S_1 + S_2$ where

$$S_2[\mathcal{A}] = -\kappa \int \mathrm{d}\mu \; \log\det \left((\bar{\partial} + \mathcal{A})_{L(x_-, \widetilde{\theta})} \right) \tag{3.4}$$

where κ is a coupling constant (later to be identified with g^2_{YM}). The fibre of $\mathbb{PT}^{3|4} \to \mathbb{E}_-^{4|8}$ over a point $(x_-, \widetilde{\theta}) \in \mathbb{E}_-^{4|8}$ is a $\mathbb{CP}^{1|0}$ that we denote by $L(x_-, \widetilde{\theta})$. In forming (3.4) we first restrict the Cauchy-Riemann operator $\bar{\partial}_{\mathcal{A}}$ to $L(x_-, \widetilde{\theta})$, then construct the determinant of this operator and finally integrate the logarithm of this determinant over the space of lines $\mathbb{CP}^{1|0} \hookrightarrow \mathbb{PT}^{3|4}$ (as discussed in section 2, this is antichiral superspace) using the measure

$$\mathrm{d}\mu := \frac{1}{4!} \epsilon_{abcd} \, \mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \mathrm{d}x^c \wedge \mathrm{d}x^d \; \mathrm{d}^8\widetilde{\theta}. \tag{3.5}$$

Note that because we are already integrating over all $\widetilde{\theta}$s, $\mathrm{d}x_-$ may be equated with $\mathrm{d}x$.

The determinant of a $\bar{\partial}$-operator is not really a function, but a section of a line bundle over the space of connections as discussed by Quillen [23]. The line bundle can be provided with a metric and a connection, but in general these may not be flat, and the bundle itself could be non-trivial. However, this line bundle must be trivial over the space of $\mathbb{CP}^{1|0}$ fibres since this is antichiral Euclidean space $\mathbb{E}_-^{4|8}$, which doesn't have sufficient topology. (The line bundle would similarly be trivial over $S^{4|8}$ but we would have to argue more carefully for more complicated spacetimes.) We still have the freedom to choose a trivialisation, but this freedom can be reduced somewhat by following an observation of Quillen that, by picking a base-point $\mathcal{A}_0$ in the space of connections, we can adjust the metric on the determinant line bundle using the norm of $\mathcal{A} - \mathcal{A}_0$ on the $\mathbb{CP}^1$ so that the associated Chern connection is flat. Thus we can trivialize the bundle (up to a constant) using this flat connection once we have picked a base-point in the space of connections. In the associated flat frame we are justified in treating the determinant naïvely as a function and may integrate its logarithm over $\mathbb{E}^{4|8}$. However, as is familiar in physics from anomaly calculations, the dependence of this trivialization on a fixed background connection $\mathcal{A}_0$, which can be taken to be flat, nevertheless breaks gauge invariance. Indeed, under (3.3) the determinant varies as

$$\det(\overline{\partial} + \mathcal{A})_L \to \exp\left(\frac{1}{2\pi}\int_L g^{-1}\partial g \wedge \mathcal{A}\right)\det(\overline{\partial} + \mathcal{A})_L. \tag{3.6}$$

In [6] the determinant's lack of gauge invariance was cited as additional evidence for coupling between the open and closed sectors of the twistor-string, leading to conformal gravity, as gauge invariance may be restored by a compensating transformation of the B-field in the closed string sector. Here though, we are concerned with $\log\det\overline{\partial}_{\mathcal{A}}|_L$ which under gauge transformations acquires an additive piece integrated over one copy of the fibre L. Since this term is at most quintic[3] in $\widetilde{\theta}$, it will not survive the Berezinian integration in $\mathrm{d}\mu$. Hence the full action is gauge invariant without recourse to the closed string sector.

The real justification for our action is that it is precisely equivalent to $\mathcal{N} = 4$ SYM in spacetime, as we shall soon make clear. However, as a preliminary check notice that S_1 and S_2 each contain only single-trace interactions (recall that $\log\det M = \operatorname{tr}\log M$) and that the $\mathrm{PGL}(4|4,\mathbb{C})$ transformation $[Z^\alpha, \psi^i] \mapsto [Z^\alpha, r\psi^i]$ with $r \in \mathbb{C}^*$ induces the transformations $S_1 \mapsto r^{-4}S_1$ and $S_2 \mapsto r^{-8}S_2$ (because $\psi^i \mapsto r\psi^i$ with $\pi_{A'}$ invariant implies $\widetilde{\theta}^{A'i} \mapsto r\widetilde{\theta}^{A'i}$). Although S_2 is non-polynomial in $\mathcal{A}$, this scaling together with the induced action $\lambda \mapsto r^{-1}\lambda$, $\phi \mapsto r^{-2}\phi$, $\chi \mapsto r^{-3}\chi$ and $b \mapsto r^{-4}b$ shows immediately that these component fields can only appear with certain powers. In particular, S_2 is at most quadratic in b. Moreover, following Witten's reasoning, the partition function $\mathcal{Z}(\hbar,\kappa) = \int \mathrm{D}\mathcal{A}\,\mathrm{e}^{-S[\mathcal{A}]/\hbar}$ can be made invariant under the r-scaling if we declare that $\hbar \mapsto r^{-4}\hbar$ and $\kappa \mapsto r^4\kappa$. Conservation of the associated charge then demands that an l-loop contribution to an amplitude with n_λ, n_ϕ, n_χ and n_b external fields of types λ, ϕ, χ and b respectively must scale like $\hbar^{l-1}\kappa^d$ where

$$4d = 4n_b + 3n_\chi + 2n_\phi + n_\lambda + 4(l-1). \tag{3.7}$$

In particular, this implies that all amplitudes vanish unless $3n_\chi + 2n_\phi + n_\lambda = 4m$ with m a non-negative integer. As discussed in [1], this is exactly the behaviour of the spacetime $\mathcal{N} = 4$ action.

[3]It is possible that ∂g introduces a $\widetilde{\theta}$ that is independent of the combination $\psi^i = \widetilde{\theta}^{A'i}\pi_{A'}$.

3.2 Equivalence to $\mathcal{N}=4$ SYM on spacetime

Let us now validate our claim that $S[\mathcal{A}]$ is equivalent to $\mathcal{N}=4$ SYM on spacetime. To begin, we must partially gauge-fix $\mathcal{A}$ to remove the extra symmetry beyond the spacetime gauge group. To achieve this, expand $\mathcal{A}$ in the basis (2.10) as $\mathcal{A}=\bar{e}^0\mathcal{A}_0+\bar{e}^A\mathcal{A}_A$ and impose the gauge condition

$$\overline{\partial}^*_L\mathcal{A}_0=0 \tag{3.8}$$

on all fibres. The notation $\overline{\partial}_L$ means the $\overline{\partial}$-operator on the $\mathbb{CP}^1$ fibre labelled by $L(x_-,\widetilde{\theta})$ so we are requiring that $\mathcal{A}_0$ be fibrewise co-closed with respect to the standard Fubini-Study metric of each $\mathbb{CP}^1$. Because $\mathcal{A}$ is holomorphic in ψ, for (3.8) to hold for all $\widetilde{\theta}$, it must hold for each component of the ψ-expansion separately, so $\overline{\partial}^*_L a_0=\overline{\partial}^*_L\lambda_{i0}=\overline{\partial}^*_L\phi_{ij\,0}=\overline{\partial}^*_L\chi^i_0=\overline{\partial}^*_L b_0=0$. Since the fields are (0,1)-forms and $\dim_{\mathbb{C}}L=1$, they are all $\overline{\partial}$-closed automatically. Also requiring them to be co-closed ensures that they harmonic along the fibres, so Hodge's theorem tells us that in this gauge, the restriction of the component fields to the fibres are $\mathrm{End}(E)$-valued elements of $H^1(\mathbb{CP}^1,\mathcal{O}(n))$ where n runs from 0 to -4. However, $H^1(\mathbb{CP}^1,\mathcal{O})=H^1(\mathbb{CP}^1,\mathcal{O}(-1))=0$ so that $a_0=\lambda_{i0}=0$, while the other fields may be put in standard form [19]

$$\begin{aligned}
a&=\bar{e}^A a_A(x,\pi,\hat{\pi}) & \lambda_i&=\bar{e}^A\lambda_{i\,A}(x,\pi,\hat{\pi})\\
\phi_{ij}&=\bar{e}^0\Phi_{ij}(x)+\bar{e}^A\phi_{ij\,A}(x,\pi,\hat{\pi}) & \chi^i&=2\bar{e}^0\frac{\widetilde{\Lambda}^i_{A'}(x)\hat{\pi}^{A'}}{\pi\cdot\hat{\pi}}+\bar{e}^A\chi^i_A(x,\pi,\hat{\pi})\\
b&=3\bar{e}^0\frac{B_{A'B'}(x)\hat{\pi}^{A'}\hat{\pi}^{B'}}{(\pi\cdot\hat{\pi})^2}+\bar{e}^A b_A(x,\pi,\hat{\pi})
\end{aligned} \tag{3.9}$$

where, as indicated, Φ, $\widetilde{\Lambda}$ and B depend only on x and numerical factors are included in the definition of $\widetilde{\Lambda}$ and B for later convenience. The $\mathcal{A}_A$ components are as yet unconstrained. The non-trivial step in achieving this gauge choice is in setting $a_0=0$, which implies that we have found a frame for E that is holomorphic up the fibres of $\mathbb{PT}^{3|4}\to\mathbb{E}^{3|4}$. This requires that the bundle E is trivial up the fibres (a standard assumption of the Ward construction) which is not guaranteed in general, but, in a perturbative context will follow from a smallness assumption on a_0.

We have only restricted $\mathcal{A}$ to be fibrewise harmonic, rather than harmonic on all of twistor space, so there is some residual gauge freedom: any gauge transformation by h satisfying $\overline{\partial}^*_L\overline{\partial}_L h=0$ for all fibres L leaves (3.8) unchanged. If $\overline{\partial}^*_L\overline{\partial}_L h=0$, h is a globally defined solution to the Laplacian of weight zero on a $\mathbb{CP}^1$, so is constant on each fibre by the maximum principle. Hence $h=h(x)$ only and the residual gauge freedom is precisely by spacetime gauge transformations.

To impose our gauge choice in the path integral, we should include ghosts $c\in\Omega^{(0,0)}_{\mathbb{PT}^{3|4}}(\mathrm{End}E)$ and antighosts $\bar{c}\in\Omega^{(0,3)}_{\mathbb{PT}^{3|4}}(\mathrm{End}E)$ together with a Nakanishi-Lautrup field $m=[Q,\bar{c}]\in\Omega^{(0,3)}_{\mathbb{PT}^{3|4}}(\mathrm{End}E)$ where Q is the BRST operator. The gauge-fixing term

$$\int\Omega\,\mathrm{d}^4\psi\wedge\mathrm{tr}\left[Q,\bar{c}\left(\overline{\partial}^*_L\mathcal{A}_0\right)\right]=\int\Omega\,\mathrm{d}^4\psi\wedge\mathrm{tr}\left\{m\left(\overline{\partial}^*_L\mathcal{A}_0\right)+\bar{c}\overline{\partial}^*_L\left[\overline{\partial}_0+\mathcal{A}_0,c\right]\right\} \tag{3.10}$$

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imposes (3.8) upon integrating out m, whereupon $\mathcal{A}_0 \sim (\psi)^2$ as we have seen. The ghost kinetic term $\bar{c}\overline{\partial}^*\overline{\partial}c$ only involves terms whose coefficient fields have net r-charge -4 in an expansion of $\bar{c}c$, while the ghost-matter interaction $\bar{c}[\mathcal{A}, c]$ picks out terms of net r-charge -2, -1 and 0 from the $\bar{c}c$ expansion. Considering the $(\bar{c}, c)$ couplings as a 4×4 matrix with rows and columns labelled by r-charge, we see that this matrix has upper-triangular form, with ghost-matter mixing occuring only off the diagonal. Hence the Fadeev-Popov determinant is independent of $\mathcal{A}$ and the ghost sector decouples from the path integral.

In terms of the expansion (3.9), the Chern-Simons part of the action becomes

$$\begin{aligned} S_1[\mathcal{A}] = \frac{\mathrm{i}}{2\pi}\int \frac{\Omega\wedge\bar{\Omega}}{(\pi\cdot\hat{\pi})^4}\,\mathrm{tr}\Bigg\{ & 3\frac{B_{A'B'}\hat{\pi}^{A'}\hat{\pi}^{B'}}{(\pi\cdot\hat{\pi})^2}\left(\pi^{C'}\frac{\partial a^A}{\partial x^{AC'}} + \frac{1}{2}[a^A, a_A]\right) \\ & + 2\frac{\widetilde{\Lambda}^i_{A'}\hat{\pi}^{A'}}{\pi\cdot\hat{\pi}}\left(\pi^{B'}\frac{\partial\lambda_i^A}{\partial x^{AB'}} + [a^A, \lambda_{iA}]\right) + \frac{\epsilon^{ijkl}}{4}\Phi_{ij}\left(\pi^{A'}\frac{\partial\phi_{kl}^A}{\partial x^{AA'}} + [a^A, \phi_{kl\,A}]\right) \\ & + \frac{\epsilon^{ijkl}}{2}\Phi_{ij}\lambda_k^A\lambda_{lA} + \left(b^A\overline{\partial}_0 a_A + \chi^{iA}\overline{\partial}_0\lambda_{iA} + \frac{\epsilon^{ijkl}}{8}\phi_{ij}^A\overline{\partial}_0\phi_{kl\,A}\right)\Bigg\} \end{aligned} \tag{3.11}$$

where we note that the expression inside the braces is weightless. Since S_2 is independent of $\mathcal{A}_A\bar{e}^A$ while the untilded fields b_A and χ^i_A appear here only linearly (in the last line of the above formula), they play the role of Lagrange multipliers. Integrating them out of the partition function enforces $\bar{e}^0\overline{\partial}_0 a_A = \bar{e}^0\overline{\partial}_0\lambda_{i\,A} = 0$ and so a_A and $\lambda_{i\,A}$ must be holomorphic in π. Since they have holomorphic weights $+1$ and 0 respectively, we find

$$a_A(x, \pi, \hat{\pi}) = A_{AA'}(x)\pi^{A'} \qquad \text{and} \qquad \lambda_{i\,A}(x, \pi, \hat{\pi}) = \Lambda_{i\,A}(x) \tag{3.12}$$

for some spacetime dependent fields $A_{AA'}$ and $\Lambda_{i\,A}$. Similarly, because $\phi_{ij\,A}$ appears only quadratically it may be eliminated[4] using its equation of motion

$$\overline{\partial}_0\phi_{ij\,A} = \pi^{A'}\left(\frac{\partial\Phi_{ij}}{\partial x^{AA'}} + [A_{AA'}, \Phi_{ij}]\right) \tag{3.13}$$

where we have used (3.12). This implies

$$\phi_{ij\,A} = \frac{1}{\pi\cdot\hat{\pi}}\hat{\pi}^{A'}D_{AA'}\Phi_{ij} \tag{3.14}$$

where $D_{AA'}$ is the usual spacetime gauge covariant derivative and, as in (3.9), Φ depends only on spacetime coordinates. Inserting our expressions for a_A, λ_A and ϕ_A into (3.11) now reduces the action to

$$\begin{aligned} S_1[\mathcal{A}] = \frac{\mathrm{i}}{2\pi}\int \frac{\Omega\wedge\bar{\Omega}}{(\pi\cdot\hat{\pi})^4}\,\mathrm{tr}\Bigg\{ & \frac{3}{2}B_{A'B'}F_{C'D'}\frac{\hat{\pi}^{A'}\hat{\pi}^{B'}\pi^{C'}\pi^{D'}}{(\pi\cdot\hat{\pi})^2} + 2\widetilde{\Lambda}^i{}_{A'}D^B{}_{B'}\Lambda_{iB}\frac{\hat{\pi}^{A'}\pi^{B'}}{\pi\cdot\hat{\pi}} \\ & + \frac{\epsilon^{ijkl}}{8}\Phi_{ij}D^A{}_{A'}D_{AB'}\Phi_{kl}\frac{\hat{\pi}^{A'}\pi^{B'}}{\pi\cdot\hat{\pi}} + \frac{\epsilon^{ijkl}}{2}\Phi_{ij}\Lambda_i^A\Lambda_{lA}\Bigg\} \end{aligned} \tag{3.15}$$

[4]At the cost of a field-independent determinant.

where $F^{A'B'}$ is the selfdual part of the curvature of $A_{AA'}$. None of the remaining fields depend on π or $\hat{\pi}$, so we can integrate out the fibres (see the appendix for details). Doing so, one finds

$$S_1[\mathcal{A}] = \int \mathrm{d}^4x \ \mathrm{tr}\left\{\frac{1}{2}B_{A'B'}F^{A'B'} + \widetilde{\Lambda}^i_{A'}D^{AA'}\Lambda_{iA} + \frac{\epsilon^{ijkl}}{16}D^A_{A'}\Phi_{ij}D^{A'}_A\Phi_{kl} + \frac{\epsilon^{ijkl}}{2}\Phi_{ij}\Lambda^A_k\Lambda_{lA}\right\} \tag{3.16}$$

and, as is familiar from Witten's work [1], holomorphic Chern-Simons theory on $\mathbb{PT}^{3|4}$ thereby reproduces the anti-selfdual interactions of $\mathcal{N} = 4$ SYM in an action first discussed by Chalmers & Siegel [24].

We must now find the contribution from S_2 and to do so, we must vary the determinant. The formula for the variation follows from the prescription given earlier; we do not wish to give the full theory here, but refer the reader to the discussion in section 3 of [25]. The device of renormalizing the metric on the Quillen determinant line bundle was not used in [25], but it simply has the effect of removing the appearance of α^* from equation 3.3 of that paper (α^* is $\mathcal{A}^*$ in our notation, with the $*$ denoting complex conjugation with respect to a chosen Hermitian structure on E). On restricting to gauge group SU(N), we obtain

$$\delta \log\det\left(\overline{\partial}_{\mathcal{A}}\big|_L\right) = \int_L \mathrm{tr}\, J\delta\mathcal{A} \tag{3.17}$$

where

$$J(\pi_1) = \lim_{\pi_1\to\pi_2}\left(G(\pi_1,\pi_2) - \frac{1}{2\pi\mathrm{i}}\frac{I}{\pi_1\cdot\pi_2}\right)\pi_1\cdot\mathrm{d}\pi_1 \tag{3.18}$$

in which π_1, π_2 are abbreviations for the homogeneous coordinates $\pi_{1A'}$ on L_1 *etc.*, G is the Greens function for $\overline{\partial}_{\mathcal{A}}$ on sections of E of weight -1 over L, I is the identity matrix and π denotes the usual ratio of the circumference to the diameter of a circle. Using the relation

$$\delta G(\pi_1,\pi_2) = -\int_L G(\pi_1,\pi_3)\delta\mathcal{A}(x,\widetilde{\theta},\pi_3)G(\pi_3,\pi_2)\,\pi_3\cdot\mathrm{d}\pi_3\,, \tag{3.19}$$

we can expand S_2 in powers of $\mathcal{A}$ as

$$\log\det\left(\overline{\partial}_{\mathcal{A}}\big|_L\right) = \mathrm{tr}\left\{\ln\overline{\partial}_L + \sum_{r=1}^{\infty}\frac{1}{r}\left(\frac{-1}{2\pi\mathrm{i}}\right)^r\int\frac{\pi_1\cdot\mathrm{d}\pi_1}{\pi_r\cdot\pi_1}\mathcal{A}_1\frac{\pi_2\cdot\mathrm{d}\pi_2}{\pi_1\cdot\pi_2}\mathcal{A}_2\cdots\frac{\pi_r\cdot\mathrm{d}\pi_r}{\pi_{r-1}\cdot\pi_r}\mathcal{A}_r\right\} \tag{3.20}$$

where $(1/2\pi\mathrm{i})(1/\pi_i\cdot\pi_j)$ is the Green's function at $\mathcal{A} = 0$[5] for the $\overline{\partial}_L$-operator on $L = \mathbb{CP}^1$. Each $\mathcal{A}$ in this expansion is restricted to lie on a copy of the fibre over the same point $(x,\widetilde{\theta})$ in spacetime. In particular, they each depend on the same $\widetilde{\theta}^{A'i}$ so because $\psi^i = \widetilde{\theta}^{A'i}\pi_{A'}$ and $\mathcal{A}_0 \sim (\psi)^2$ in this gauge, the series vanishes after the fourth term. Furthermore, the measure $\mathrm{d}\mu$ involves an integration $\mathrm{d}^8\widetilde{\theta}$, so we only need keep the terms proportional to $(\widetilde{\theta})^8$. Schematically then, the only relevant terms are B^2, $\Phi\widetilde{\Lambda}^2$, $\Phi^2 B$ and Φ^4. In fact, since $B_{A'B'}$ represents a selfdual 2-form on spacetime, the $\Phi^2 B$ term may also be neglected since

[5] In the gauge (3.8), the connection is trivial along the fibres, so End(E)-valued fields may be integrated over these fibres without worrying about parallel propagation. We apologize for the proliferation of πs in our Green's function, and hope the meaning is clear!

there is no way for it to form a non-vanishing scalar once we integrate out the $\mathbb{CP}^1$ fibre. The B^2 term is

$$-\kappa \int \mathrm{d}\mu \, \frac{1}{2} \left(\frac{3}{2\pi \mathrm{i}} \right)^2 \int \prod_{r=1}^{2} \frac{K_r}{\pi_r \cdot \pi_{r+1}} \mathrm{tr} \left\{ \frac{B_{A'B'} \hat{\pi}_1^{A'} \hat{\pi}_1^{B'}}{(\pi_1 \cdot \hat{\pi}_1)^2} \frac{B_{C'D'} \hat{\pi}_2^{C'} \hat{\pi}_2^{D'}}{(\pi_2 \cdot \hat{\pi}_2)^2} (\psi_1)^4 (\psi_2)^4 \right\}, \tag{3.21}$$

where we have defined the Kähler form

$$K = \frac{\pi \cdot \mathrm{d}\pi \wedge \hat{\pi} \cdot \mathrm{d}\hat{\pi}}{(\pi \cdot \hat{\pi})^2} \tag{3.22}$$

on each copy of the $\mathbb{CP}^1$ fibre. The $\widetilde{\theta}$ integrations may be evaluated straightforwardly using Nair's lemma

$$\int \mathrm{d}^8 \widetilde{\theta} \; (\psi_1)^4 (\psi_2)^4 \big|_{L(x_-, \widetilde{\theta})} = (\pi_1 \cdot \pi_2)^4 \,, \tag{3.23}$$

while the results in the appendix then allow us to integrate out the fibres in equation (3.21), yielding a contribution $-\frac{\kappa}{2} \int \mathrm{d}^4 x \; \mathrm{tr} 2 B_{A'B'} B^{A'B'}$ on spacetime. To find the contributions from the $\Phi \widetilde{\Lambda}^2$ term

$$-\kappa \int \mathrm{d}\mu \; \frac{2}{(2\pi \mathrm{i})^3} \int \prod_{r=1}^{3} \frac{K_r}{\pi_r \cdot \pi_{r+1}} \mathrm{tr} \left\{ \psi_1^i \psi_1^j \Phi_{ij} \epsilon_{klmn} \frac{\psi_2^k \psi_2^l \psi_2^m}{3!} \frac{\widetilde{\Lambda}^n_{A'} \hat{\pi}_2^{A'}}{\pi_2 \cdot \hat{\pi}_2} \epsilon_{pqrs} \frac{\psi_3^p \psi_3^q \psi_3^r}{3!} \frac{\widetilde{\Lambda}^s_{B'} \hat{\pi}_3^{B'}}{\pi_3 \cdot \hat{\pi}_3} \right\} \tag{3.24}$$

and the Φ^4 term

$$-\kappa \int \mathrm{d}\mu \; \frac{1}{4} \frac{1}{(2\pi \mathrm{i})^4} \int \prod_{r=1}^{4} \frac{K_r}{\pi_r \cdot \pi_{r+1}} \psi_1^i \psi_1^j \psi_2^k \psi_2^l \psi_3^m \psi_3^n \psi_4^p \psi_4^q \; \frac{1}{2^4} \mathrm{tr} \left\{ \Phi_{ij} \Phi_{kl} \Phi_{mn} \Phi_{pq} \right\} \tag{3.25}$$

it is helpful to first integrate out the first copy of the fibre from (3.24) and (say) the first and third copies from (3.25) using

$$\int K_1 \frac{\pi_{1A'} \pi_{1B'}}{\pi_1 \cdot \pi_2 \, \pi_3 \cdot \pi_1} \widetilde{\theta}^{iA'} \widetilde{\theta}^{jB'} = -2\pi \mathrm{i} \, \frac{\pi_{2A'} \pi_{3B'} + \pi_{3A'} \pi_{2B'}}{(\pi_2 \cdot \pi_3)^2} \widetilde{\theta}^{iA'} \widetilde{\theta}^{jB'} \,. \tag{3.26}$$

These integrations reduce the $\widetilde{\theta}$ dependence of (3.24) and (3.25) to the same form as in (3.21); integrating out these $\widetilde{\theta}$s allows us to perform the remaining fibre integrals as before. Combining all the terms, we find that the $\log \det \overline{\partial}_{\mathcal{A}}$ term provides a contribution

$$S_2[\mathcal{A}] = -\kappa \int \mathrm{d}^4 x \, \mathrm{tr} \left\{ \frac{1}{2} B_{A'B'} B^{A'B'} + \frac{1}{2} \Phi_{ij} \widetilde{\Lambda}^i_{A'} \widetilde{\Lambda}^{j\,A'} + \frac{1}{16} \epsilon^{iklm} \epsilon^{jnpq} \Phi_{ij} \Phi_{kl} \Phi_{mn} \Phi_{pq} \right\}. \tag{3.27}$$

Adding this to the Chern-Simons contribution in equation (3.16) gives the complete $\mathcal{N} = 4$ SYM action (up to the topological invariant $\mathrm{c}_2(F)$); to put it in standard form one integrates out $B_{A'B'}$, identifies $\kappa = g^2_{\mathrm{YM}}$ and rescales $\widetilde{\Lambda}_{A'} \to \widetilde{\Lambda}_{A'} / \sqrt{g_{\mathrm{YM}}}$, $\Lambda_A \to \sqrt{g_{\mathrm{YM}}} \Lambda_A$.

3.3 The MHV formalism

One of the pleasing features of the twistor action is that it provides a simple way to understand the MHV diagram formalism of Cachazo, Svrček & Witten [9]. Instead of working in the gauge (3.8), one picks an arbitrary spinor η^A and imposes the axial-like condition $\eta^A\overline{\partial}_A\lrcorner\mathcal{A}=0$. In this gauge, the $\mathcal{A}^3$ vertex of the Chern-Simons theory vanishes. However, we no longer have the restriction that $\mathcal{A}_0\sim(\psi)^2$, so the expansion

$$\log\det\left(\overline{\partial}_{\mathcal{A}}\big|_L\right)=\mathrm{tr}\left\{\ln\overline{\partial}_L+\sum_{r=1}^{\infty}\frac{1}{r}\left(\frac{-1}{2\pi\mathrm{i}}\right)^r\int\frac{\pi_1\cdot\mathrm{d}\pi_1}{\pi_r\cdot\pi_1}\mathcal{A}_1\frac{\pi_2\cdot\mathrm{d}\pi_2}{\pi_1\cdot\pi_2}\mathcal{A}_2\cdots\frac{\pi_r\cdot\mathrm{d}\pi_r}{\pi_{r-1}\cdot\pi_r}\mathcal{A}_r\right\} \tag{3.28}$$

in S_2 does not terminate. Focussing on the spin 1 sector, the action contains an infinite series of vertices each of which is quadratic in B (so as to survive the $\widetilde{\theta}$ integration) and it is easy to see that these are exactly the MHV vertices. Also, this gauge brings the substantial simplification that the only non-vanishing components of on-shell fields are $\mathcal{A}_0$. For momentum eigenstates, the $\mathcal{A}_0$ have delta function dependence on $\pi_{A'}$ supported where π'_A is proportional to the corresponding spinor part of the spacetime momentum as in [1]. We have undertaken a study of perturbation theory using this form of the action, and will present our results in a companion paper [15].

4. Theories with less supersymmetry

Having dealt with the maximally supersymmetric gauge theory, let us now study theories with $\mathcal{N}=1$ & 2 sets of spacetime supercharges. Rather than work on weighted projective spaces, our strategy here is to obtain (the SYM sector) of these theories by breaking the U(4) R-symmetry of the $\mathcal{N}=4$ theory. We will then see how to couple these SYM theories to matter in an arbitrary representation of the gauge group.

The $\mathcal{N}=4$ theory possesses a U(4) R-symmetry which, in the twistorial representation, arises from the freedom to rotate ψs into one another using the generators $\psi^i\partial/\partial\psi^j$. To reach a theory with only $\mathcal{N}=2$ supersymmetry one arbitrarily singles out two ψ directions, say ψ^3 and ψ^4, and demand that all fields depend on them only via the combination $\psi^3\psi^4$ i.e. we require invariance under the R-symmetry SU(2) in (ψ^3,ψ^4). With this restriction, the $\mathcal{N}=4$ multiplet (3.1) becomes

$$\begin{aligned}\mathcal{A}&=a+\psi^a\lambda_a+\frac{1}{2}\epsilon_{ab}\psi^a\psi^b\phi+\psi^3\psi^4\left(\widetilde{\phi}+\psi^a\chi_a+\frac{1}{2}\epsilon_{ab}\psi^a\psi^b b\right)\\&=\mathcal{A}^{(2)}+\psi^3\psi^4\mathcal{B}^{(2)}\end{aligned} \tag{4.1}$$

where a,b run from 1 to 2, and $\mathcal{A}^{(2)}$ and $\mathcal{B}^{(2)}$ have the exact field content of an $\mathcal{N}=2$ gauge multiplet and its CPT conjugate. Upon integrating out $\psi^3\psi^4$, the action S_1+S_2 becomes (dropping the wedges)

$$\begin{aligned}S_{\mathrm{gauge}}[\mathcal{A}^{(2)},\mathcal{B}^{(2)}]=&\frac{\mathrm{i}}{2\pi}\int\Omega\mathrm{d}^2\psi\,\mathrm{tr}\,\mathcal{B}^{(2)}\mathcal{F}^{(2)}\\&+\frac{\kappa}{8\pi^2}\int\mathrm{d}^4x\mathrm{d}^4\widetilde{\theta}(\pi_1\cdot\pi_2)^2\mathrm{tr}\left\{(\overline{\partial}+\mathcal{A}^{(2)})^{-1}_{21}\mathcal{B}^{(2)}_1\,(\overline{\partial}+\mathcal{A}^{(2)})^{-1}_{12}\mathcal{B}^{(2)}_2\right\}\end{aligned} \tag{4.2}$$

where $\mathcal{F}^{(2)}=\overline{\partial}\mathcal{A}^{(2)}+[\mathcal{A}^{(2)},\mathcal{A}^{(2)}]$ is the curvature of $\mathcal{A}^{(2)}$. The definition (4.1) implies that $\mathcal{B}^{(2)}$ has holomorphic weight -2 so that this action is well-defined on the projective space. The integrand in the second term of this action is understood to be restricted to copies of the $\mathbb{CP}^1$ fibres over $(x_-,\widetilde{\theta})$ as in section 3. The subscripts on the $\mathcal{B}$ fields and the Green's functions in this term label copies of the fibres, while $(\overline{\partial}+\mathcal{A})^{-1}_{ij}$ is understood to involve an integral over fibre j. Keeping only the appropriate components of the fields, it is straightforward to verify that (4.2) reproduces the standard $\mathcal{N}=2$ spacetime SYM action (up to a non-perturbative term) when the gauge (3.8) is imposed.

Notice that this method of restricting the dependence of $\mathcal{A}$ on the fermionic coordinates is similar to, but distinct from, working on a weighted projective superspace. Although $\psi^3\psi^4$ is a nilpotent object of weight 2, it is bosonic and we would not have obtained the above action from a string theory on the weighted Calabi-Yau supermanifold $\mathbb{WCP}^{3|3}(1,1,1,1|1,1,2)$. It is also interesting to consider the effect of the scaling $\psi\mapsto r\psi$. The action (4.2) is invariant under the U(1) (really, $\mathbb{C}^*$) part of the remaining U(2) R-symmetry if we shift the charge of $\mathcal{B}^{(2)}$ so that $\psi^a\mapsto r\psi^a$ induces $\mathcal{B}^{(2)}\mapsto r^2\mathcal{B}^{(2)}$. The component fields $\{a,\lambda_a,\phi\}$ and $\{\widetilde{\phi},\chi_a,b\}$ then have charges $\{0,-1,-2\}$ and $\{2,1,0\}$ respectively, exactly the grading of these fields that is familiar from Donaldson-Witten theory, for example.

Similarly, to obtain $\mathcal{N}=1$ SYM one demands that $\mathcal{A}$ depends on ψ^2, ψ^3 and ψ^4 only through the combination $\psi^2\psi^3\psi^4$ so that, calling $\psi^1=\psi$,

$$\begin{aligned}\mathcal{A}&=a+\psi\lambda-\psi^2\psi^3\psi^4(\chi+\psi b)\\&=\mathcal{A}^{(1)}-\psi^2\psi^3\psi^4\mathcal{B}^{(1)}\end{aligned}\tag{4.3}$$

with $\mathcal{A}^{(1)}$ and $\mathcal{B}^{(1)}$ containing exactly the field content of an $\mathcal{N}=1$ gauge multiplet and its CPT conjugate. The constraint that $\psi^2\psi^3\psi^4$ always appear together leaves no room for ϕ, and the action is simply

$$\begin{aligned}S_{\text{gauge}}[\mathcal{A}^{(1)},\mathcal{B}^{(1)}]=&\frac{\mathrm{i}}{2\pi}\int\Omega\mathrm{d}\psi\,\mathrm{tr}\,\mathcal{B}^{(1)}\mathcal{F}^{(1)}\\&+\frac{\kappa}{8\pi^2}\int\mathrm{d}^4x\mathrm{d}^2\widetilde{\theta}(\pi_1\cdot\pi_2)^3\mathrm{tr}\left\{(\overline{\partial}+\mathcal{A}^{(1)})^{-1}_{21}\mathcal{B}^{(1)}_1(\overline{\partial}+\mathcal{A}^{(1)})^{-1}_{12}\mathcal{B}^{(1)}_2\right\}.\end{aligned}\tag{4.4}$$

In this case, in spacetime gauge only the B^2 term survives from S_2, since all others involved ϕ. Again, it is straightforward to check that this gauge choice yields exactly the usual $\mathcal{N}=1$ action, and that the residual r-scaling is just the usual U(1) R-symmetry.

4.1 Matter multiplets

In theories with $\mathcal{N}<4$ supersymmetries, additional multiplets are possible. At $\mathcal{N}=2$ there is a hypermultiplet consisting of fields with helicities $(-\frac{1}{2}^1,0^2,+\frac{1}{2}^1)$ together with its CPT conjugate, where the superscripts denote multiplicity. At $\mathcal{N}=1$ we have a chiral multiplet whose component fields have helicities $(-\frac{1}{2}^1,0^1)$ together with its antichiral CPT conjugate. These multiplets were first constructed in twistor superspaces by Ferber [18] and take the forms

$$\mathcal{N}=2\text{ hyper}\begin{cases}\mathcal{H}=\rho+\psi^a h_a+\dfrac{\epsilon_{ab}}{2}\psi^a\psi^b\widetilde{\mu}\\\widetilde{\mathcal{H}}=\mu+\psi^a\widetilde{h}_a+\dfrac{\epsilon_{ab}}{2}\psi^a\psi^b\widetilde{\rho}\end{cases}\tag{4.5}$$

where $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are each fermionic and have weight -1, and

$$\mathcal{N}=1 \text{ chiral} \begin{cases} \mathcal{C}=\nu+\psi m \\ \widetilde{\mathcal{C}}=\widetilde{m}+\psi\widetilde{\nu} \end{cases} \tag{4.6}$$

where $\mathcal{C}$ is fermionic and of weight -1, while $\widetilde{\mathcal{C}}$ is bosonic and of weight -2; all the above fields are (0,1)-forms. The matter fields may take values in arbitrary representations R of the gauge group. Their actions take similar forms, for example

$$\begin{aligned} S_{\text{hyp}}[\mathcal{H},\widetilde{\mathcal{H}},\mathcal{A}^{(2)}] = & \int \Omega\, \mathrm{d}^2\psi\, \mathrm{tr}\left\{\widetilde{\mathcal{H}}\,\overline{\partial}_{\mathcal{A}^{(2)}}\mathcal{H}\right\} \\ & +2\kappa\int \mathrm{d}^4x\mathrm{d}^4\widetilde{\theta}\mathrm{tr}\Big\{\left(\overline{\partial}_{\mathcal{A}^{(2)}}\right)^{-1}_{31}\mathcal{H}_1\left(\overline{\partial}_{\mathcal{A}^{(2)}}\right)^{-1}_{12}\widetilde{\mathcal{H}}_2\left(\overline{\partial}_{\mathcal{A}^{(2)}}\right)^{-1}_{23}\mathcal{B}^{(2)}_3\,\pi_1\!\cdot\!\pi_3\,\pi_2\!\cdot\!\pi_3\Big\} \\ & -\frac{3\kappa}{2}\int \mathrm{d}^4x\,\mathrm{d}^4\widetilde{\theta}\,\mathrm{tr}\Big\{\left(\overline{\partial}_{\mathcal{A}^{(2)}}\right)^{-1}_{41}\mathcal{H}_1\left(\overline{\partial}_{\mathcal{A}^{(2)}}\right)^{-1}_{12}\widetilde{\mathcal{H}}_2\left(\overline{\partial}_{\mathcal{A}^{(2)}}\right)^{-1}_{23} \\ & \qquad\qquad \times\mathcal{H}_3\left(\overline{\partial}_{\mathcal{A}^{(2)}}\right)^{-1}_{34}\widetilde{\mathcal{H}}_4\;\pi_1\cdot\pi_3\,\pi_2\cdot\pi_4\Big\} \end{aligned} \tag{4.7}$$

for a hypermultiplet in the fundamental representation and

$$\begin{aligned} S_{\text{ch}}[\mathcal{C},\widetilde{\mathcal{C}},\mathcal{A}^{(1)}] = & \int \Omega\, \mathrm{d}\psi\, \mathrm{tr}\left\{\widetilde{\mathcal{C}}\,\overline{\partial}_{\mathcal{A}^{(1)}}\mathcal{C}\right\} \\ & +2\kappa\int \mathrm{d}^4x\mathrm{d}^2\widetilde{\theta}\;\mathrm{tr}\Big\{\left(\overline{\partial}_{\mathcal{A}^{(1)}}\right)^{-1}_{31}C_1\left(\overline{\partial}_{\mathcal{A}^{(1)}}\right)^{-1}_{12}\widetilde{C}_2\left(\overline{\partial}_{\mathcal{A}^{(1)}}\right)^{-1}_{23}\mathcal{B}^{(1)}_3\,\pi_1\!\cdot\!\pi_2(\pi_2\!\cdot\!\pi_3)^2\Big\} \\ & -\frac{3\kappa}{2}\int \mathrm{d}^4x\,\mathrm{d}^2\widetilde{\theta}\;\mathrm{tr}\Big\{\left(\overline{\partial}_{\mathcal{A}^{(1)}}\right)^{-1}_{41}C_1\left(\overline{\partial}_{\mathcal{A}^{(1)}}\right)^{-1}_{12}\widetilde{C}_2\left(\overline{\partial}_{\mathcal{A}^{(1)}}\right)^{-1}_{23} \\ & \qquad\qquad \times C_3\left(\overline{\partial}_{\mathcal{A}^{(1)}}\right)^{-1}_{34}\widetilde{C}_4\;(\pi_1\cdot\pi_3)^2(\pi_2\cdot\pi_4)^2\Big\} \end{aligned} \tag{4.8}$$

for a fundamental chiral multiplet, where the traces and $\overline{\partial}_{\mathcal{A}}$-operators are in the fundamental representation. The actions are well-defined on the projective superspaces, with the weights of the measures being balanced by those of the fields. Again, the subscripts label the copy of the fibre on which the relevant field is to be evaluated, and the operators $\left(\overline{\partial}_{\mathcal{A}}\right)^{-1}_{ij}$ involve an integral over the j^{th} fibre. These actions may be obtained by symmetry reduction, using the decomposition of the $\mathcal{N}=4$ gauge multiplet into $\mathcal{N}=2$ gauge and hyper-multiplets, or $\mathcal{N}=1$ gauge and chiral multiplets, and then changing the representation (and number) of matter multiplets. In fact all these matter couplings can be obtained by an appropriate symmetry reduction from some large gauge group and the $\int \mathrm{d}^4x\mathrm{d}^8\theta$ expressions in (4.8) and (4.9) may be understood in that context as additional contributions from a 'log det' term.

Since the matter fields are all (0,1)-forms, there is an additional symmetry that may be surprising from the spacetime perspective. For example, when $\mathcal{C}$ is in the fundamental representation while $\widetilde{\mathcal{C}}$ is in the antifundamental, then the complete $\mathcal{N}=1$ action $S_{\text{gauge}}+S_{\text{ch}}$ is invariant under the usual gauge transformations

$$\begin{aligned} \overline{\partial}+\mathcal{A}^{(1)} &\to g(\overline{\partial}+\mathcal{A}^{(1)})g^{-1} & \qquad \mathcal{C} &\to g\,\mathcal{C} \\ \mathcal{B}^{(1)} &\to g\mathcal{B}^{(1)}g^{-1} & \qquad \widetilde{\mathcal{C}} &\to \widetilde{\mathcal{C}}g^{-1}, \end{aligned} \tag{4.9}$$

but it is also invariant under the transformations

$$\mathcal{C} \to \mathcal{C} + \overline{\partial}_{\mathcal{A}^{(1)}} M \qquad \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}} + \overline{\partial}_{\mathcal{A}^{(1)}} \widetilde{M} \qquad \mathcal{B}^{(1)} \to \mathcal{B}^{(1)} + \mathcal{C}\widetilde{M} - M\widetilde{\mathcal{C}} \tag{4.10}$$

where $M \in \Gamma_{\mathbb{PT}^{3|1}}(E(-1))$ is a fermion and $\widetilde{M} \in \Gamma_{\mathbb{PT}^{3|1}}(E^*(-2))$ is a boson. The fact that the matter fields are only defined up to exact forms is a direct consequence of the fact that physical information is encoded in cohomology on twistor space. To evaluate any path integral involving these matter fields, this additional symmetry needs to be fixed. In particular, requiring $\overline{\partial}^*_L \mathcal{C} = 0$ and $\overline{\partial}^*_L \widetilde{\mathcal{C}} = 0$ on each fibre L allows one to reduce the theory to (the kinetic and D-term parts of) the usual spacetime action, in exactly the same way as was done in section 3. Here, no residual freedom remains once these conditions are imposed because the fields M and $\widetilde{M}$ each have negative weight, but $H^0(\mathbb{CP}^1, \mathcal{O}(n)) = 0$ for $n < 0$.

5. Discussion

We have studied actions for twistorial gauge theories, showing how they are related to the standard spacetime and MHV formalisms. A detailed investigation of perturbation theory using this action will be presented in a companion paper [15]. However, the demonstration that $S_1 + S_2$ is perturbatively equivalent to $\mathcal{N} = 4$ SYM in spacetime at the level of the partition function makes it clear that conformal supergravity does not appear in our treatment.

It is instructive to contrast our picture with the original twistor-string proposal. Scattering amplitudes between states with wavefunctions $\mathcal{A}_1, \ldots, \mathcal{A}_n$ may be obtained in any quantum field theory by varying the generating functional

$$\mathrm{e}^{F[\mathcal{A}_{\mathrm{cl}}]} = \int_{\mathcal{A} \to \mathcal{A}_{\mathrm{cl}}} \mathrm{D}\mathcal{A}\ \mathrm{e}^{-S[\mathcal{A}]} \tag{5.1}$$

with respect to $\mathcal{A}_{\mathrm{cl}}$ in the directions $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and evaluating at $\mathcal{A}_{\mathrm{cl}} = 0$, where the path integral in (5.1) is taken over field configurations that approach $\mathcal{A}_{\mathrm{cl}}$ asymptotically. Witten conjectured [26] that the free energy for twistor-strings could be evaluated as

$$\mathrm{e}^{F[\mathcal{A}_{\mathrm{cl}}]} = \sum_{g=0,d=1}^{\infty} \kappa^d \int_{\mathcal{M}^{\mathrm{conn}}_{g,d}} \mathrm{d}\mu_{g,d}\ \det\left(\overline{\partial}_{\mathcal{A}_{\mathrm{cl}}}\big|_{C'}\right) \tag{5.2}$$

where $\mathcal{M}^{\mathrm{conn}}_{g,d}$ is a contour in the moduli space of connected, genus g degree d curves in $\mathbb{PT}^{3|4}$, $\mathrm{d}\mu_{g,d}$ is some measure on $\mathcal{M}^{\mathrm{conn}}_{g,d}$ and $C' \in \mathcal{M}^{\mathrm{conn}}_{g,d}$. In the disconnected prescription, this conjecture may be recast as

$$\mathrm{e}^{F[\mathcal{A}_{\mathrm{cl}}]} = \int_{\mathcal{A} \to \mathcal{A}_{\mathrm{cl}}} \mathrm{D}\mathcal{A}\ \mathrm{e}^{-S_1[\mathcal{A}]} \left\{ \sum_{d=1}^{\infty} \kappa^d \int_{\mathcal{M}_d} \mathrm{d}\mu_d\ \det\left(\overline{\partial}_{\mathcal{A}}\big|_C\right) \right\} \tag{5.3}$$

as was argued in [14]. Here, $S_1[\mathcal{A}]$ is the holomorphic Chern-Simons action, $\mathcal{M}_d$ is a contour in the moduli space of maximally disconnected, genus zero degree d curves in $\mathbb{PT}^{3|4}$ and $C \in \mathcal{M}_d$. In this formula, the effect of the functional integral is to introduce

Chern-Simons propagators connecting the different disconnected components of the curves together. In [27] Gukov, Motl & Neitzke argued that these two formulations of twistor-string theory could be shown to be equivalent by deforming the contour in the space of curves through regions in which components of the disconnected curves come together in such a way as to eliminate the Chern-Simons propagators and connect the curves.

We can obtain an effective action that would lead to Witten's conjecture as follows. First, choose the contour $\mathcal{M}_d$ to be $(\mathbb{E}^{4|8})^d/\mathrm{Sym}_d$, the set of unordered d-tuples $(x_i, \widetilde{\theta}_i)$, $i = 1, \ldots, d$ of (possibly coincident) points in $\mathbb{E}^{4|8}$, so that the degree d curve C is the union $C = \cup_{i=1}^d L(x_i, \widetilde{\theta}_i)$ of d lines. On a disconnected curve, the determinant factorizes to give

$$\det\left(\overline{\partial}_{\mathcal{A}}\big|_C\right) = \prod_{i=1}^{d} \det\left(\overline{\partial}_{\mathcal{A}}\big|_{L(x_i,\widetilde{\theta}_i)}\right) . \tag{5.4}$$

Similarly, the measure $\mathrm{d}\mu_d$ on $\mathcal{M}_d$ can be written $\mathrm{d}\mu_d = \prod_{i=1}^d \mathrm{d}^4 x_i \mathrm{d}^8 \widetilde{\theta}_i$. Putting this together, we find that the conjecture implies

$$\begin{aligned} \mathrm{e}^{F[\mathcal{A}_{\mathrm{cl}}]} &= \int \mathrm{D}\mathcal{A}\, \mathrm{e}^{-S_1[\mathcal{A}]} \left\{ \sum_{d=1}^{\infty} \frac{\kappa^d}{d!} \int_{(\mathbb{E}^{4|8})^d} \prod_{i=1}^{d} \mathrm{d}^4 x_i \mathrm{d}^8 \widetilde{\theta}_i \ \det \overline{\partial}_{\mathcal{A}}\big|_{L(x_i,\widetilde{\theta}_i)} \right\} \\ &= \int \mathrm{D}\mathcal{A}\, \mathrm{e}^{-S_1[\mathcal{A}]} \left\{ \sum_{d=1}^{\infty} \frac{1}{d!} \left(\kappa \int_{\mathbb{E}^{4|8}} \mathrm{d}^4 x \mathrm{d}^8 \widetilde{\theta} \ \det \overline{\partial}_{\mathcal{A}}\big|_{L(x,\widetilde{\theta})} \right)^d \right\} \\ &= \int \mathrm{D}\mathcal{A}\ \exp\left(-S_1[\mathcal{A}] + \kappa \int_{\mathbb{E}^{4|8}} \mathrm{d}^4 x \mathrm{d}^8 \widetilde{\theta} \ \det \overline{\partial}_{\mathcal{A}}\big|_{L(x,\widetilde{\theta})} \right) , \end{aligned} \tag{5.5}$$

where the $1/d!$ factors take account of the Sym_d in the definition of $\mathcal{M}_d$. Thus, instead of the $S_2[\mathcal{A}] = -\kappa \int \log\det\overline{\partial}_{\mathcal{A}}$ in our theory, the twistor-string inspired conjecture would seem to require the different action $\widetilde{S}_2[\mathcal{A}] = -\kappa \int \det\overline{\partial}_{\mathcal{A}}$. However, expanding $\widetilde{S}_2[\mathcal{A}]$ in $\mathcal{A}$ shows that this latter form contains spurious multi-trace terms, so these are present in the original twistor-string proposal even at the level of the action. Moreover, $\widetilde{S}_2[\mathcal{A}]$ is not gauge invariant because of the behaviour of the determinant discussed in section 3. Restoration of gauge invariance can only be achieved at the cost of coupling to the closed string sector. As we have emphasized, the action of section 3 possesses neither of these unwelcome features.

It is of course of great interest to see whether the action of section 3 can be given a string interpretation and a 'connected prescription' found. While we do not yet have a full understanding of this, the following remarks may be of interest. The natural observables of real Chern-Simons theory on a three manifold (say S^3) are the Wilson loops $W_R(\gamma) = \mathrm{tr}_R \mathrm{P}\exp\oint_\gamma A$ depending on some representation R of the gauge group. The correlation function [28]

$$\langle \prod W_{R_i}(\gamma_i)\rangle = \int \mathrm{D}A\ \exp\left(\frac{1}{4\pi}\int_{S^3} \mathrm{tr}\left\{ A\mathrm{d}A + \frac{2}{3}A^3 \right\}\right) \prod W_{R_i}(\gamma_i) \tag{5.6}$$

computes link invariants of the curves $\gamma_i \subset S^3$ depending on representations R_i. The Chern-Simons theory on S^3 may be interpreted as the open string field theory of the A-model on T^*S^3 [4] and the Wilson loops themselves find a stringy interpretation in terms of

Lagrangian branes $L_i \subset T^*S^3$ with $L_i \cap S^3 = \gamma_i$. The field theory on the worldvolume of a single such brane wrapping L_i contains a complex scalar in an N-dimensional representation R of the gauge group of the Chern-Simons theory on the S^3. Integrating out this scalar produces $\det^{(R)} \mathrm{d}_A|_{\gamma_i}$. This determinant may be related to the holonomy around γ_i by the formula

$$\det{}^{(R)}\mathrm{d}_A|_\gamma = \det\left(1 - \left(\mathrm{P}\exp\oint_\gamma A\right)_R\right) \tag{5.7}$$

which follows from ζ-function regularization (see e.g. [29]). Hence, the Chern-Simons expectation value

$$\langle \det \mathrm{d}_A|_\gamma \rangle = \int \mathrm{D}A \; \exp\left(-S_{\mathrm{CS}}[A] + \log\det\left(1 - \mathrm{P}\exp\oint_\gamma A\right)\right) \tag{5.8}$$

may be viewed as a generating functional for all the observables associated to the knot γ [30] upon expanding in powers of the holonomy. Notice that the effective action here is $\log\det \mathrm{d_A}$.

The partition function we presented in section 3 may be formally understood to arise as the expectation value of an infinite product of determinants in the holomorphic Chern-Simons theory

$$\int \mathrm{D}\mathcal{A}\; \mathrm{e}^{-S_1[\mathcal{A}]} \prod_{(x_-,\widetilde{\theta})} \det \overline{\partial}_{\mathcal{A}}\big|_{L(x_-,\widetilde{\theta})} = \int \mathrm{D}\mathcal{A}\; \mathrm{e}^{-S_1[\mathcal{A}]} \exp\left(\int_{\mathbb{E}_-^{4|8}} \mathrm{d}\mu\; \log\det \overline{\partial}_{\mathcal{A}}\big|_{L(x_-,\widetilde{\theta})}\right), \tag{5.9}$$

so it is tempting to interpret this as the generating functional for all observables associated to all possible degree 1 holomorphic curves in $\mathbb{PT}^{3|4}$. In searching for a string interpretation, we would like to find objects which support only certain types of amplitude, graded by d as in (3.7). To this end, one might seek an analogue of knot invariants for holomorphic curves. Holomorphic linking has been far less studied than real linking (see [31, 32] for the Abelian case), but it may be exactly what is needed here (see [33, 34] for earlier discussions of holomorphic linking in twistor space). In order to study link invariants in the real category, one needs to supply framings both of the underlying 3-manifold M and of the knots $\gamma_i \in M$, and from the Chern-Simons or A-model point of view, framings arise via a coupling to the gravitational or closed string sector. However, the expectation value (5.6) depends on the choice of framing only through a simple phase, and it is perfectly possible to make sense of link invariants nonetheless. One might hope that the closed string sector of the twistor-string is no more harmful.

In our view, it seems as though the ingredients of twistor-string theory are correct — perhaps only the recipe needs adjusting. We hope that the considerations we have presented in this paper will help to illuminate the story further.

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A. Integrating over the fibres

In showing that our twistor actions reduce to spactime ones, it is necessary to integrate over the $\mathbb{CP}^1$ fibres of $\mathbb{PT} \to \mathbb{E}$. Specifically, in equation (3.15) we needed to integrate expressions of the generic type

$$\int \frac{\Omega \wedge \bar{\Omega}}{(\pi \cdot \hat{\pi})^4} S_{A'B'\ldots} T_{C'D'\ldots} \frac{\pi^{A'}\pi^{B'} \cdots \hat{\pi}^{C'}\hat{\pi}^{D'} \cdots}{(\pi \cdot \hat{\pi})^n} \tag{A.1}$$

where S, T are spacetime dependent tensors with n indices each; in fact, in all the cases that arise in this paper, $S \in \mathrm{Sym}_n\, \mathbb{S}^+$. We start by noting that this integral is well-defined on the projective twistor space, and hence on each $\mathbb{CP}^1$ fibre. From (2.10)–(2.12) we have

$$\begin{aligned} \frac{\Omega \wedge \bar{\Omega}}{(\pi \cdot \hat{\pi})^4} &= \frac{\pi \cdot \mathrm{d}\pi \wedge \hat{\pi} \cdot \mathrm{d}\hat{\pi}}{(\pi \cdot \hat{\pi})^4} \wedge \left(\mathrm{d}x^{AA'} \wedge \mathrm{d}x^{BB'} \wedge \mathrm{d}x^{CC'} \wedge \mathrm{d}x^{DD'}\right) \pi_{A'}\pi_{B'}\hat{\pi}_{C'}\hat{\pi}_{D'}\epsilon_{AB}\epsilon_{CD} \\ &= \mathrm{d}^4x \frac{\pi \cdot \mathrm{d}\pi \wedge \hat{\pi} \cdot \mathrm{d}\hat{\pi}}{(\pi \cdot \hat{\pi})^2}. \end{aligned} \tag{A.2}$$

where we have used $\epsilon^{abcd} = \epsilon^{AD}\epsilon^{BC}\epsilon^{A'C'}\epsilon^{B'D'} - \epsilon^{AC}\epsilon^{BD}\epsilon^{A'D'}\epsilon^{B'C'}$ and we remind the reader that our σ-matrices are normalized so that $\sigma^2 = 1$. Hence (A.1) becomes

$$\int_{\mathbb{E}} \mathrm{d}^4x \int_{\mathbb{CP}^1} \frac{\pi \cdot \mathrm{d}\pi \wedge \hat{\pi} \cdot \mathrm{d}\hat{\pi}}{(\pi \cdot \hat{\pi})^{n+2}} S_{A'B'\ldots} T_{C'D'\ldots} \pi^{A'}\pi^{B'} \cdots \hat{\pi}^{C'}\hat{\pi}^{D'} \cdots \tag{A.3}$$

which is the also form that arises when reducing $S_2[\mathcal{A}]$ to spacetime. An object $S := S_{A'B'\ldots}\pi^{A'}\pi^{B'}\cdots$ with n πs is annihilated by the $\overline{\partial}$-operator on the $\mathbb{CP}^1$ and hence is an element of $H^0(\mathbb{CP}^1, \mathcal{O}(n))$. On the other hand,

$$T := T_{C'D'\ldots} \frac{\hat{\pi}^{C'}\hat{\pi}^{D'}\cdots}{(\pi \cdot \hat{\pi})^{n+2}} \hat{\pi} \cdot \mathrm{d}\hat{\pi} \tag{A.4}$$

is also $\overline{\partial}$-closed (since $\dim_{\mathbb{C}} \mathbb{CP}^1 = 1$) and is in fact harmonic [19] (indeed, we used this in the text to solve the gauge condition (3.8)). Thus it represents an element of $H^1(\mathbb{CP}^1, \mathcal{O}(-n-2))$. Serre duality asserts that

$$H^1(\mathbb{CP}^1, \mathcal{O}(-n-2) \simeq H^0(\mathbb{CP}^1, \Omega^1(\mathcal{O}(n+2))) \tag{A.5}$$

and in our case the duality pairing is given by

$$\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{CP}^1} (\pi \cdot \mathrm{d}\pi\ S) \wedge T = -\frac{1}{n+1} S_{A'B'\ldots} T^{A'B'\ldots} \tag{A.6}$$

which is straightforward to check explicitly by working in local coordinates on the $\mathbb{CP}^1$. Hence we find

$$\int_{\mathbb{PT}} \frac{\Omega \wedge \bar{\Omega}}{(\pi \cdot \hat{\pi})^4} S_{A'B'\ldots} T_{C'D'\ldots} \frac{\pi^{A'}\pi^{B'} \cdots \hat{\pi}^{C'}\hat{\pi}^{D'} \cdots}{(\pi \cdot \hat{\pi})^n} = -\frac{2\pi\mathrm{i}}{(n+1)} \int_{\mathbb{E}} \mathrm{d}^4x\ S_{A'B'\ldots} T^{A'B'\ldots} \tag{A.7}$$

which was used in section 3.

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